Testing Consumer Rationality using Perfect Graphs and Oriented Discs

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Commodity Market

Consider an agent in an n-commodity market:

- Each commodity has price/unit p_i
- Market price vector $\mathbf{p} \in \mathbf{R}^n_+$
- Agent requests bundle $\mathbf{x}^* \in \mathbf{R}^n_+$

Multiple observations at multiple price-points

$$\text{Data-set} = \{(\mathbf{p}_1, \mathbf{x}_1), (\mathbf{p}_2, \mathbf{x}_2), \ldots, (\mathbf{p}_m, \mathbf{x}_m)\}$$

At prices $\mathbf{p}_i$, agent demands bundle $\mathbf{x}_i$, for $i = 1$ to m.

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Revealed Preference

Consider two pairs, $(\mathbf{p}_i, \mathbf{x}_i)$ and $(\mathbf{p}_j, \mathbf{x}_j)$

- If $\mathbf{p}_j \cdot \mathbf{x}_i \leq \mathbf{p}_j \cdot \mathbf{x}_j$,
- then $\mathbf{x}_i$ was more affordable than $\mathbf{x}_j$ at prices $\mathbf{p}_j$. $\Rightarrow \quad \mathbf{x}_j \succeq \mathbf{x}_i$

Demanded bundle is **revealed preferred** to every more-affordable bundle.

Note. Geometrically, $\mathbf{x}_j$ revealed preferred to every point below the hyperplane due to $\mathbf{p}_j$

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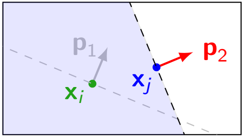

Possible Configurations

The possible geometric relations between two items fall in 3 categories:

No relation

One preferred

Mutual Preference

Can we determine if preference data is consistent?

- Clearly a 2-cycle ($\mathbf{a} \succ \mathbf{b} \succ \mathbf{a}$) is invalid. .... Non truthful
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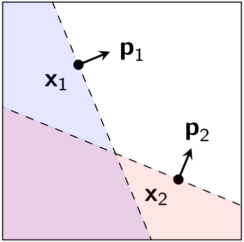

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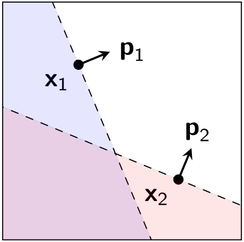

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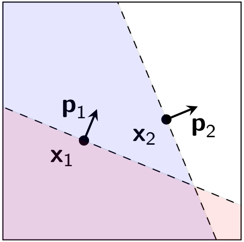

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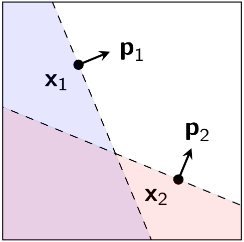

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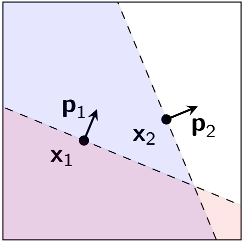

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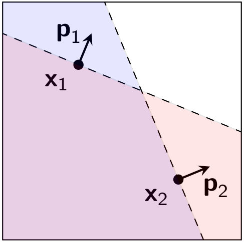

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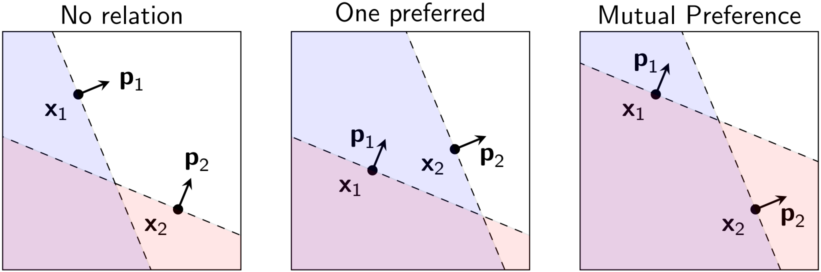

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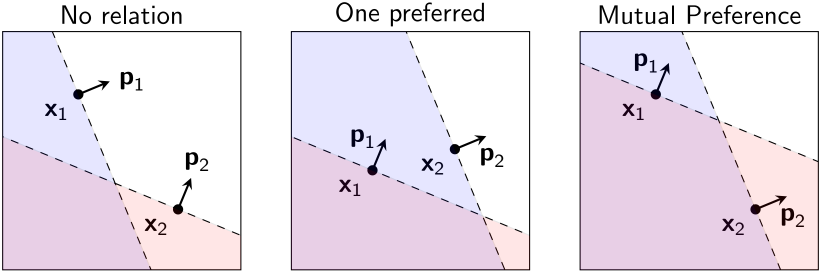

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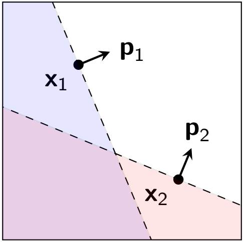

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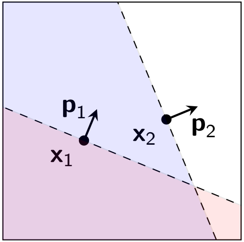

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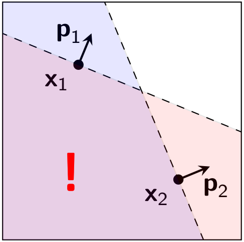

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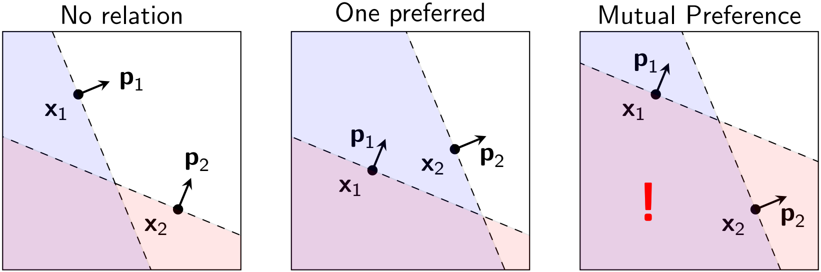

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We can build a **Preference (di-)Graph** to check for cycles

- Vertex set is data: $(\mathbf{p}_1, \mathbf{x}_1), (\mathbf{p}_2, \mathbf{x}_2), \ldots, (\mathbf{p}_m, \mathbf{x}_m)$
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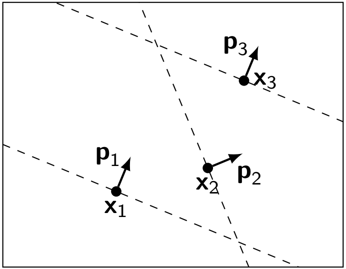

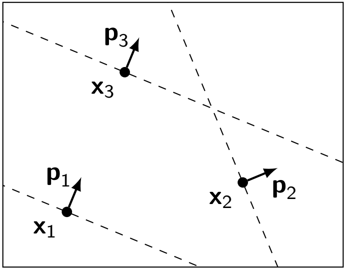

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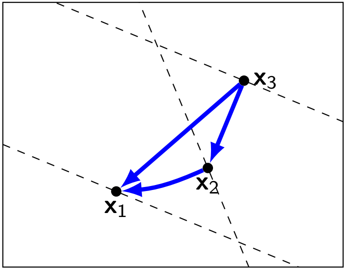

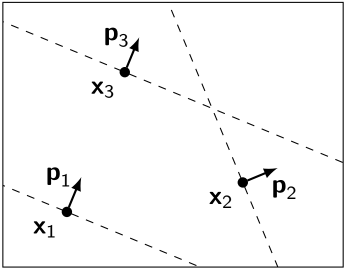

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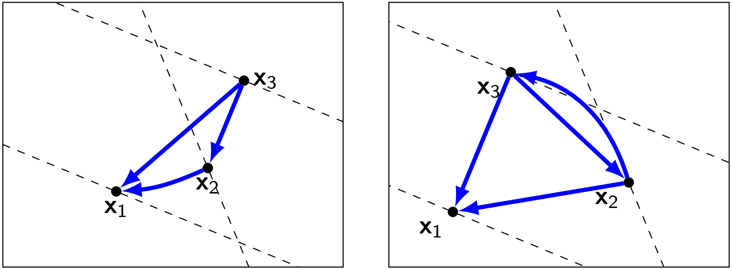

Degree of Rationality

[Afriat, '67] The agent is rational $\iff$ preference graph is acyclic.

We ask: *how* rational is agent?

Definition. An agent's *degree of rationality* is the least number of data pairs to ignore for the agent to seem rational. Rational *iff* DoR $= 0$.

Note 1. This was defined by Houtman and Maks in '88.

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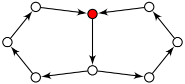

Degree of Rationality, Complexity

What is the complexity of MFVS in preference graphs?

- If market size (n) large, then any graph is a preference graph

$$p^i_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \qquad x^i_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } (i, j) \in D \\ 2 & \text{if } (i, j) \notin D \end{cases}$$

Our Results

- **Theorem 1.** If $n = 2$, *Degree of Rationality* is in P
- **Theorem 2.** If $n \geq 3$, *DoR* decision problem is NP-complete

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Theorem 1. *Degree of rationality* is poly-time for $n = 2$.

Proof Sketch.

1. **[Rose, '58]**: If $n = 2$, every cycle contains a digon (2-cycle)
2. **Corollary.** MFVS is Min. Vertex Cover on the set of digons
3. **Lemma.** Digon graphs of pref. graphs are perfect for $n = 2$. (New class of perfect graphs?)
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[Rose, '58]: If $n = 2$, every cycle contains the vertex set of a digon.

Therefore, hitting every digon hits every cycle.

Definition. The *digon graph* of a directed graph $D = (V, A)$ is $G = (V, E)$, where E is exactly the set of digons in D.

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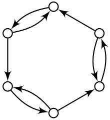

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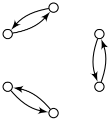

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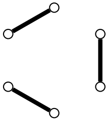

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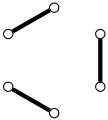
[Rose, '58]: If $n = 2$, every cycle contains the vertex set of a digon.

Therefore, hitting every digon hits every cycle.

Definition. The *digon graph* of a directed graph $D = (V, A)$ is $G = (V, E)$, where E is exactly the set of digons in D.

Example.



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Examples of Digon Graphs

Which graphs can be constructed as digon graphs? Some examples

Complete Graphs,

Complete Bip. Graphs,

Complete r-part. Graphs,

- Complements of complete r-partite graphs,
- Many small graphs,
- etc.

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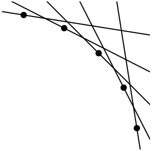

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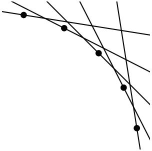

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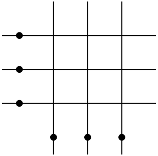

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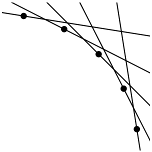

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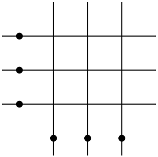

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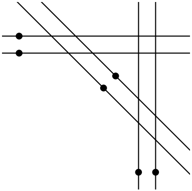

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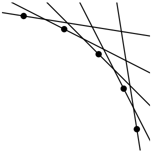

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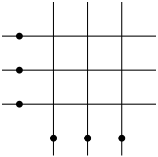

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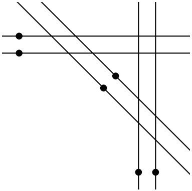

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Digon Graphs of Pref. Graphs are Perfect when $n = 2$

Sub-Lemma. If $\mathbf{x} \lessgtr \mathbf{y}$, $\mathbf{x} \lessgtr \mathbf{z}$ and $\mathbf{y} \not\lessgtr \mathbf{z}$, then either

Proof: Suppose not,

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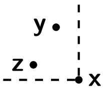

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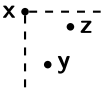

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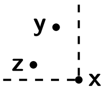

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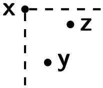

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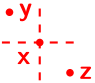

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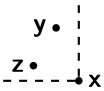

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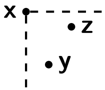

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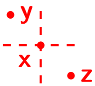

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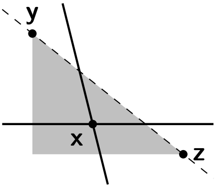

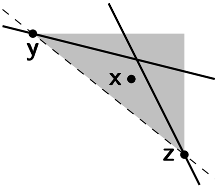

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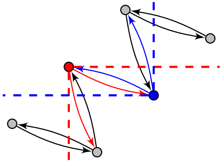
Corollary. Long paths "alternate" in the digon graph



Corollary.

1. The digon graph contains no **odd** holes on ≥ 5 vertices.
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[CRST, '06] Strong Perfect Graph Theorem

A graph is perfect $\iff$ contains no odd holes and no odd antiholes.

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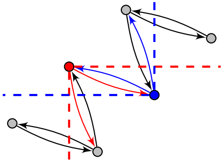

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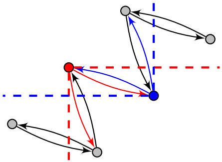

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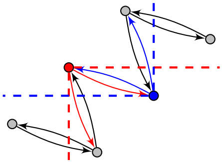

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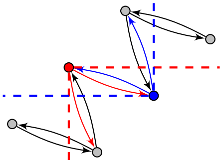

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Digon Graphs as a Class of Perfect Graphs

Open problem:

Characterize the class of digon graphs of pref. graphs for $n = 2$.

We know:

- DG $\nleq$ Complements of bipartite graphs
- DG $\nleq$ Complements of line gr. of bipartite graphs
- DG $\not\subset$ Bipartite graphs
- DG $\not\subset$ Line gr. of bipartite graphs

And maybe DG $\subsetneq$ Comparability graphs

3-Commodity Markets

Theorem 2. The decision problem is NP-complete for $n \geq 3$.

Proof Sketch.

1. **[WK, '88]**: Reduce planar 3-SAT to vert. cover on "gadget graphs"
2. **Remark.** Can reduce vert. cover to MFVS on a graph of digons.
3. **Lemma.** This graph of digons is an "oriented-disk graph"
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[WK, '88]: Reduce planar 3-SAT to vertex cover on "gadget graphs":

- Add a triangle per clause, labelled with variable type
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- Connect each clause to the cycles, opposite label.

Example. Graph for $\varphi = (\bar{u}_1 \vee u_2 \vee u_3) \wedge (u_1 \vee \bar{u}_2 \vee u_3)$

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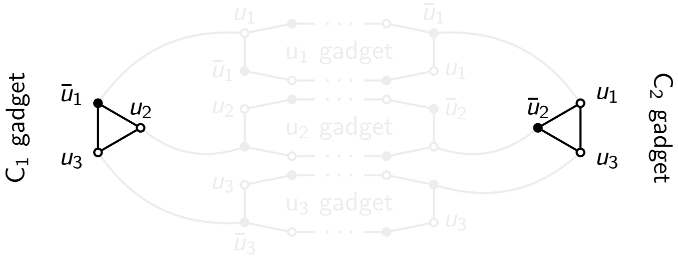

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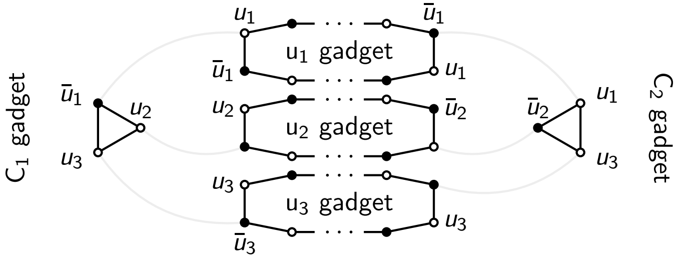

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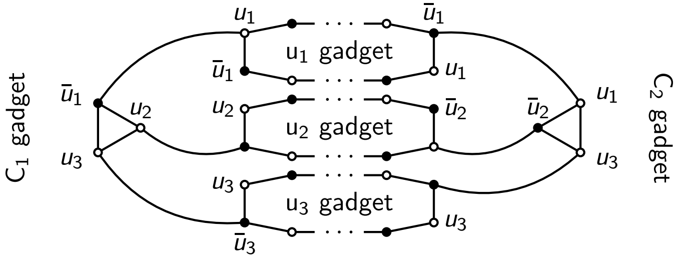

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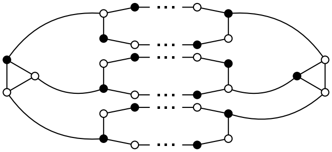

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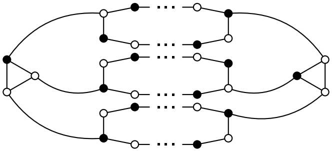

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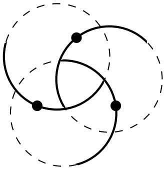

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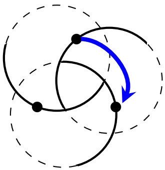

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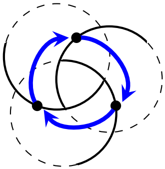

Vertex Cover Reduces to FVS on a Digon Graph

Remark.

- Take any undirected graph, and replace each edge with a digon.
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- Min. Vertex Cover on the underlying graph is equivalent to MFVS on this digraph. (Any new cycle passes through a digon)

Claim. There exists an **oriented-disk graph** which is a graph of digons for the gadget graph, plus an acyclic set of edges.

This would give a reduction from planar 3-SAT to MFVS on ODG's.

Clause Gadgets, Long Paths and Cycles

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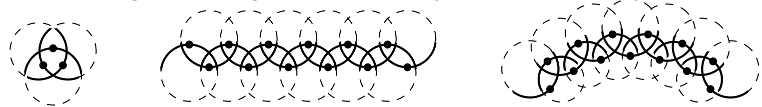
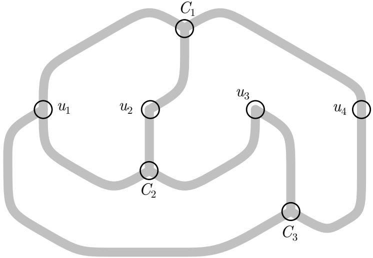


The Gadget Graph is Planar

What does this oriented-disk drawing look like?

1. Take a planar drawing of the variable-clause graph.
2. Replace the clause vertices with clause gadgets.*
3. Trace around the edges incident to each variable with its cycle.
4. * **How do we connect the cycles to the clause gadgets?**



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The Gadget Graph is Planar

What does this oriented-disk drawing look like?

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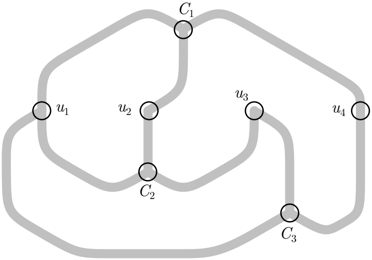

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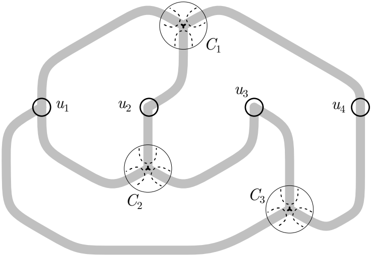

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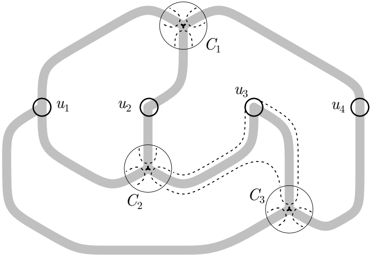

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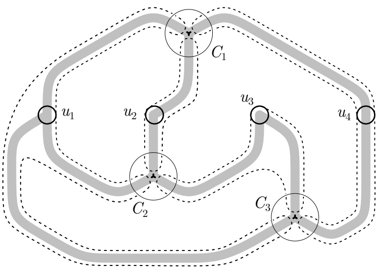

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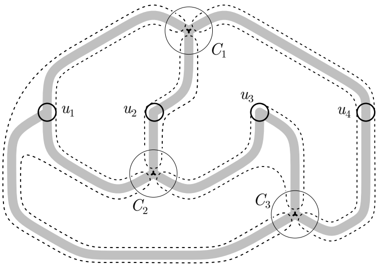

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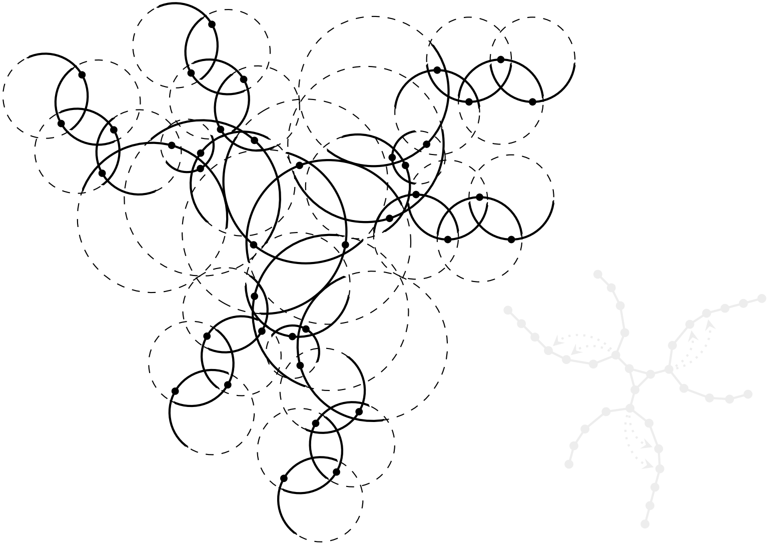

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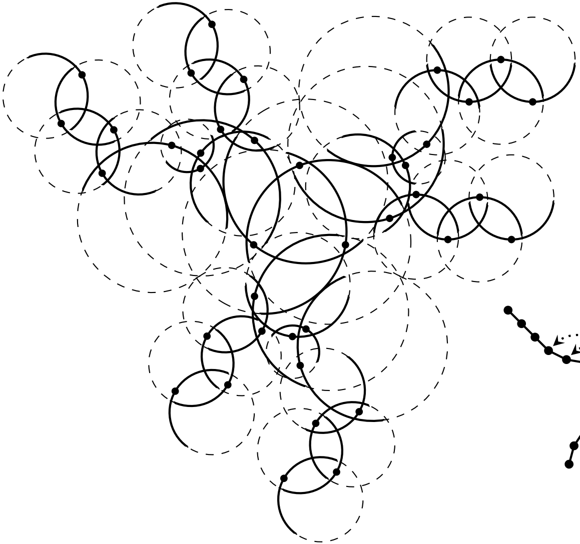

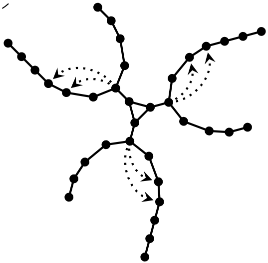

3-Commodity Markets

Theorem 2. The decision problem is NP-complete for $n \geq 3$.

Proof Sketch.

1. **[WK, '88]**: Reduce planar 3-SAT to vert. cover on "gadget graphs"
2. **Remark.** Can reduce vert. cover to MFVS on a graph of digons.
3. **Lemma.** This graph of digons is an "oriented-disk graph"
4. **Lemma.** Any ODG is a valid preference graph for $n = 3$.

So **Theorem 2.** follows as a corollary. □

Lemma. Oriented-disk graphs are valid preference graphs for $n = 3$.

Can map any planar drawing onto a small sphere section

Map points onto sphere, cut through sphere with plane to get disks (Price vector is plane normal)

Vertex revealed preferred to any point inside disk

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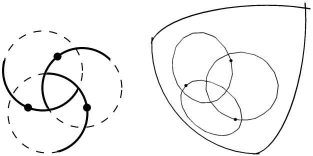

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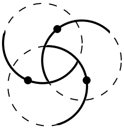

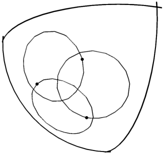

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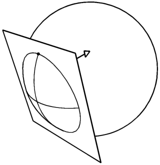

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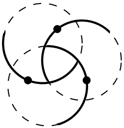

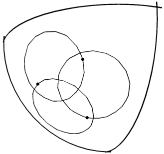

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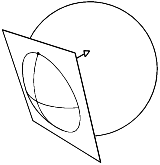

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Further Research

We have shown that $n = 2$ is the threshold for poly-time solvability

- What about approximation complexity?
- Can we provide combinatorial algorithm for $n = 2$?
- Can we characterise preference graphs for fixed n?
- Does bounding DoR affect welfare in a combinatorial auction? (RP constraints are used to impose truthfulness.)

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